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1997 J. Phys. A: Math. Gen. 30 4665

(http://iopscience.iop.org/0305-4470/30/13/017)

J. Phys. A: Math. Gen. **30** (1997) 4665–4675. Printed in the UK PII: S0305-4470(97)74123-4

Maxwell's square roots from the metric tensors of wave surfaces and branches of solutions of the photon and phonon wave equations

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Received 1 May 1996, in final form 17 December 1996

Abstract. For the wave equations of optics and acoustics of isotropic media the infinite families of three-dimensional plane wave Cauchy operators are found by a direct tensor method. These families form involutive Lie groups. Their generators N can be found by taking square roots from the unit tensors of the wavefront subspaces with outer normal $\boldsymbol{n}$. In optics the basic structural elements of N are complex involutive operators (reflection isometries) described by pairs of complex vectors $\boldsymbol{S}$ and $\boldsymbol{C}$, which satisfy the metric condition $\boldsymbol{S} \cdot \boldsymbol{C} = 1$ ($\boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0$), and also by a pair of projective operators $\pm(1 - \boldsymbol{n} \otimes \boldsymbol{n})$ of the two-dimensional space of a plane orthogonal to $\boldsymbol{n}$. In the acoustics of isotropic media, in view of the inequality of the longitudinal and transverse wave velocities, the generators N are represented as a linear combination of an involutive operator and a diad $\boldsymbol{n} \otimes \boldsymbol{n}$. It is shown that the projection of the average energy flux $\langle \boldsymbol{P} \rangle \boldsymbol{n}$ of the wave is conserved in the general case $N^+ \neq N$. The families of vectors $\boldsymbol{S} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \mathrm{i}\alpha \boldsymbol{e}_2)$, $\boldsymbol{C} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + (\mathrm{i}/\alpha)\boldsymbol{e}_2)$, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$, $\boldsymbol{e}_1 = \boldsymbol{e}_1^*$, $\boldsymbol{e}_2 = \boldsymbol{e}_2^*$, $\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 1$, $\alpha = \alpha^*$ being a part of N, are indicated. For these families the global operators $\exp[\mathrm{i}kN(z - z_0)]$ acting on initial-field vectors give states described by the right-hand and left-hand elliptical helices. The wave normal $\boldsymbol{n}$ characterizes the direction of the angular momentum of the field and for the case $\alpha = 1$ turns out to be equivalent to the Darboux vector known in geometry.

1. Introduction

Faraday's and Maxwell's geometrical ideas [1, 2] have received development from investigations in electrodynamics, gravitation and elementary particles [3–11]. The works of Gauss, Laplace, Poisson, Stokes, Green, Listing, Riemann, Clifford, Lie, Klein, Beltrami, Lobachevskii, Ricci, Levi-Civita, Darboux, Cartan, Weyl and other mathematicians became the base for development of modern topological methods and algebraic geometry. Close connections between Maxwell's equations and geometrical constructions of the Riemann space [8, 11–14] were established. The fact of the topological multiconnectivity of the three-dimensional space noted by Maxwell [2] was used in Einstein–Rainich–Misner–Wheeler geometrodynamics [10, 13]. This dynamics gives, in particular, a purely geometrical description of electromagnetism. A charge is interpreted with the help of electromagnetic fields without sources which yield to Maxwell's equations for the empty space. Maxwell's legacy has stimulated the development of gauge Yang–Mills fields, Higgs fields, topological phases (Berry's phases) and the discovery of the photon bandgap structure [10, 11, 14, 15]. Hertz's words confirm that the essence of Maxwell theory is concentrated in the system of

0305-4470/97/134665+11$19.50

his equations. In modern spin physics these equations are applicable to massless photons and they replace the Dirac equation which describes the behaviour of particles having a rest mass [7, 9]. In [7] a few types of photon propagator were considered and this pointed to the ambiguity of the choice of these propagators determined by gauge. It is also important to take into account the fact that each plane wave solution of Maxwell's equations is necessarily completely polarized in a transverse direction. A non-polarized wave cannot be a solution of Maxwell's equations [16]. Therefore in three-dimensional space a two-dimensional subspace—a wave surface—turns out to be outstanding. The orientation of such a surface immersed in the three-dimensional space is given by a wave normal $\boldsymbol{n}$.

In differential geometry there are two approaches to the description of properties of curved two-dimensional surfaces immersed in three-dimensional space. The first is associated with an investigation of the surface by the observer located in the enveloped three-dimensional space (external approach). The second approach is connected with the supposition that the observer is located on the surface and does not know about the existence of the third dimension (internal approach). In the latter case the internal geometry of the 'two-dimensional' observer in general differs from the Euclidean geometry. In particular, the observer notes that the sum of the interior angles of a triangle is not equal to π. Nevertheless, this observer can completely describe in enough detail the geometry of the surface introducing into consideration the metric tensor of this surface.

The fundamental fact of electromagnetic theory is that electromagnetic waves are transverse. In this connection the wave equations of optics have some specific nature in comparison with other wave equations (for example, with the equations of acoustics). Using the first, external approach we have to take material tensors (e.g. the dielectric permittivity tensor ε) as tensors of three-dimensional space. In some studies (see [17]) the material tensors are treated as metric tensors. But using the internal approach the material, or metric, tensors have to be taken to be 'truncated' and acting in the wavefront subspace. In this sense one can speak about the metric defined at the wave surface.

In differential geometry using the method of the Darboux–Cartan mobile basis [12] there is an analogue of the vector $\boldsymbol{n}$—a Darboux vector. A vector $\boldsymbol{n}$ appears in Fedorov's covariant (coordinate free) theory of electromagnetic and acoustic waves in anisotropic media [4, 5]. This vector is real for homogeneous waves and complex for inhomogeneous ones. In [4, 5] the operators (tensors and vectors) of the three-dimensional space were used and time was included in the relations of the theory in an asymmetrical way. Nevertheless, all relations are covariant under Lorentz transformations. Such a covariant Fedorov approach corresponds to the external method of the description of curved surfaces in differential geometry and the theory of Lie groups. Fedorov gave an external method of the description of light beam polarization with the help of a coherence tensor (beam tensor) Φ [5] determined on the $SO(3)$ group. In the tensor Φ a wave normal vector or, to be exact, an antisymmetric second rank pseudotensor $\boldsymbol{n}^{\times}$ dual to the vector $\boldsymbol{n}$ [4, 5] is included. In [18–24] a step was taken towards the external description of plane waves in complex media, enabling the avoidance of a need for the separate consideration of eigen (normal) waves in a medium, and working with complete vectors of electromagnetic fields. In our constructions the important role is played by the refractive index operator N, which is a generator of continuous one-parametric Lie groups of the evolutional solutions (photon propagators) of wave equations in complex media [18–20]. The generator N generalizes a scalar refractive index $n = \sqrt{\varepsilon\mu}$ of an isotropic medium. The index of degeneration of N is equal to two which determines the structure of the appropriate simple Lie group and group attached to it [25] (root and cyclic subspaces of propagators). Here we are going to investigate symmetry features (isometries) of the group generators of the one-dimensional solutions of wave equations of the optics and acoustics of isotropic media. It displays global topological properties of the space (Huygens principle) in which photons and phonons propagate [26]. Our consideration shows that the tensor N contains $\boldsymbol{n}$ even for isotropic media [24]. It is given by the square root $\sqrt{\varepsilon\mu}\sqrt{1-\boldsymbol{n}\otimes\boldsymbol{n}}$, the infinite set of its branches characterizing a continuous group known in mathematics as a kaleidoscopic Coxeter's group of isometries [27]. In electrodynamics the index N turns out to be a construction from the electrodynamical scalars ε, μ and from the metric tensor of the two-dimensional wavefront subspace orthogonal to $\boldsymbol{n}$. The branches of the square root $N(\varepsilon\mu)^{-1/2} = \sqrt{(1-\boldsymbol{n}\otimes\boldsymbol{n})}$ form a family of vacuum involutional operators [18]. The importance of the problem of taking a root from the unit was specified by Hamilton [28]. Descartes [29] gave a geometrical interpretation of this operation. We note that at the transition to the internal method of the description of wave surfaces in isotropic media the tensor N coincides with the differential 2×2 Jones matrix [30, 31]. Jones considered some branches of his matrix but did not associate them with metric tensors of wave surfaces. Cartan developed the calculus of the external forms and using the method of the Darboux mobile basis stated a systematic approach for the construction of various spaces.

The number of coordinate degrees of freedom of an electromagnetic field is not equal to three, but is equal to two. The Lie group arises here as an algebraic abstraction of the concept of field symmetry. Groups are also manifolds [32], for example they are curves and surfaces in three-dimensional space. They look locally like Euclidean subspaces, but globally can be absolutely different from these subspaces. Fields are usually synthesized from plane waves since any front is locally plane. For any wavenumber k_x, k_y, k_z there are two independent polarization states. In anisotropic media it is necessary to use generalized Fourier analysis, with the integration on groups of exponential evolution operators whose generators are infinitesimal operators [18, 20, 33]. The appropriate operator relations for the construction of wave beams were given in [33]. The work of [34] is closely related to our approach. In particular, the matrices M_x, M_y, M_z given there correspond to the projections of our tensor $\mathrm{i}\boldsymbol{n}^{\times}$. In [34] the polarization calculations with the help of 2×2 Jones matrices, 4×4 Müller matrices and Poincaré sphere are explained on the basis of generator representations of O_3^+, $SU(2)$ and Lorentz groups. The fundamental Maxwell equations were not, however, used in [34]. Earlier we considered the connection of the direct method based on the covariant Maxwell equations with other methods of the description of polarizations [35] and, therefore, we shall not dwell on these details. We note only that our approach enables us to carry out the Fourier synthesis of polarized fields (spatial beams in three-dimensional space) with the help of plane wave solutions and we do not have to divide the field into partial (normal) waves. The approach enables us to avoid many complications, especially in the case of the directions near the optical axis and other special directions. As was shown in [21], there are many such exotic directions in transparent biaxial crystal plates. In all cases the synthesis presumes an integration on groups of the plane wave evolution operators of three-dimensional space. It is essential that in the problem of the synthesis it is not the complete group of rotations $SO(3)$ that plays the primary role, but its subgroup of axial rotations only. This is in complete agreement with the fundamentals of quantum electrodynamics, in particular with the fact that there is no rest system for a photon and there is always an outstanding direction in the space—the direction of the photon momentum.

Special attention should be given to the remarkable properties of the equations of an electromagnetic field in the form given to them by Maxwell and Hertz. These equations contain the great algebra–geometrical 'potential'. The operator form of the notation itself 'suggests' formulation of the evolution problem in an operator way. We devote this paper to isotropic media. We consider sets of one-dimensional evolution solutions of optics and acoustics of such media. The symmetry nature of these sets is described by Lie groups. For the electromagnetic field the manifolds of elliptical helicoids correspond to these groups. To emphasize the connection of our approach with the methods of differential geometry (topology) we give general expressions for the Darboux vectors $\boldsymbol{\delta}$ of the families of elliptical helices in three-dimensional space. These vectors are used in the method of mobile Frenét trihedral [36] and are defined by curvatures and torsions of helices. We show that in the particular case of circular helices all the vectors $\boldsymbol{\delta}$, regardless of the radius and pitch of a screw, have the same constant direction coinciding with the wave normal $\boldsymbol{n}$. At the end of the paper we give involutive solutions of the wave equation of acoustics of isotropic media to ensure the possibility of comparison with appropriate solutions in electrodynamics. We hope that our consideration will help to promote a better understanding of connections between the different approaches in the description of polarization states in both optics and acoustics.

2. Involutive operators in optics of isotropic media

We proceed using Maxwell's equations

$$\nabla^{\times}\boldsymbol{E} = \mathrm{i}k\boldsymbol{B} \qquad \nabla^{\times}\boldsymbol{H} = -\mathrm{i}k\boldsymbol{D} \tag{1}$$

for the fields with time dependence $\exp(-\mathrm{i}\omega t)$, where ω is the frequency. In (1) the operator $\nabla^{\times}_{ik} = -\nabla^{\times}_{ki} \equiv e_{ilk}\nabla_l$ dual to ∇_l (e_{ilk} is the Levi-Civita pseudotensor) is introduced [4, 5], and $k = \omega/c$. For repeated indices summation is implied (Einstein's rule). Let us consider the plane wave solutions of equations (1),

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(z)\exp(-\mathrm{i}\omega t)$$

where $z = \boldsymbol{n}\cdot\boldsymbol{r}$, $\boldsymbol{n}$ is an unit vector of the wave normal (analogous expressions take place for $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$). It should be noted that $z = \boldsymbol{n}\cdot\boldsymbol{r} =$ constant is the equation of a plane in the normal form of Gess. We have $\nabla^{\times} \to \boldsymbol{n}^{\times}(\mathrm{d}/\mathrm{d}z)$ and (1) can be written in the form

$$\boldsymbol{n}^{\times}\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}z} = \mathrm{i}k\boldsymbol{B} \qquad \boldsymbol{n}^{\times}\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}z} = -\mathrm{i}k\boldsymbol{D}. \tag{2}$$

For anisotropic crystals the constitutive equations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} \qquad \boldsymbol{B} = \mu\boldsymbol{H} \tag{3}$$

are valid, where ε and μ are dielectric permittivity and magnetic permeability tensors, respectively. We restrict ourself to the consideration of isotropic media whose values of ε and μ are scalars. Eliminating the vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ from (2) and (3) we obtain

$$(\boldsymbol{n}^{\times})^2\frac{\mathrm{d}^2\boldsymbol{H}}{\mathrm{d}z^2} - k^2\varepsilon\mu\boldsymbol{H} = 0 \tag{4}$$

$-(\boldsymbol{n}^{\times})^2 = I = 1 - \boldsymbol{n}\otimes\boldsymbol{n}$ being the projective operator on planes orthogonal to $\boldsymbol{n}$, 1 being the unit tensor of the three-dimensional space. It follows immediately from (4) that $\boldsymbol{n}\cdot\boldsymbol{H} = 0$ (transversality of the field). For the tangential component $\boldsymbol{H}_\tau = I\boldsymbol{H} = -(\boldsymbol{n}^{\times})^2\boldsymbol{H}$ of the field we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2\varepsilon\mu I\right)\boldsymbol{H}_\tau = 0. \tag{5}$$

We shall search for $\boldsymbol{H}_\tau$ in the form of the following evolutional solutions (Cauchy operators)

$$\boldsymbol{H}_\tau(z) = \exp[\mathrm{i}kN(z - z_0)]\boldsymbol{H}_\tau(z_0) \tag{6}$$

where the second rank refractive index tensor N is introduced. The vector $\boldsymbol{H}_\tau(z_0)$ in a reference point z_0 is supposed to be given. Then substituting (6) in (5) and taking into account the arbitrariness of the vector $\boldsymbol{H}_\tau(z_0)$ we obtain the following tensor dispersion equation to determine N

$$N^2 = \varepsilon\mu I \tag{7}$$

where $\boldsymbol{n}N = N\boldsymbol{n} = 0$. The equation (7) generalizes [24] the well known scalar Maxwell's relation $n^2 = \varepsilon\mu$. From (7) it follows that

$$N = \sqrt{\varepsilon\mu I} = \sqrt{\varepsilon\mu(1 - \boldsymbol{n}\otimes\boldsymbol{n})}.$$

The impedance tensor γ is determined by relation $[\boldsymbol{n}\cdot\boldsymbol{E}] = \gamma\boldsymbol{H}_\tau$. From the second equation of system (1) it follows that $\boldsymbol{n}^\times \mathrm{d}\boldsymbol{H}/\mathrm{d}z = -\mathrm{i}k\varepsilon\boldsymbol{E}$, $\mathrm{d}\boldsymbol{H}_\tau/\mathrm{d}z \equiv \mathrm{i}kN\boldsymbol{H}_\tau = \mathrm{i}k\varepsilon[\boldsymbol{n}\cdot\boldsymbol{E}]$, hence

$$\gamma = \frac{1}{\varepsilon}N = \sqrt{(\mu/\varepsilon)I} = \sqrt{(\mu/\varepsilon)(1 - \boldsymbol{n}\otimes\boldsymbol{n})}. \tag{8}$$

The operator $N_0 = \sqrt{I}$ whose square is equal to the projective operator I is a part of the expressions for N and γ. In mathematics such operators are known as involutive operators [27]. The operator N is a generator of the continuous Lie group of the evolutional solutions (6), giving

$$\Omega(z - z_0) \equiv \exp[\mathrm{i}kN(z - z_0)] = \boldsymbol{n}\otimes\boldsymbol{n} + I\cos[k\sqrt{\varepsilon\mu}(z - z_0)] + \mathrm{i}N_0\sin[k\sqrt{\varepsilon\mu}(z - z_0)]. \tag{9}$$

The average energy flux of the electromagnetic field is $\langle\boldsymbol{P}\rangle = (c/8\pi)\,\mathrm{Re}[\boldsymbol{E}\cdot\boldsymbol{H}^*]$ and in view of (6), (8) and (9) its projection on the direction of the normal $\boldsymbol{n}$ is equal to

$$\begin{aligned}
\langle\boldsymbol{P}\rangle\cdot\boldsymbol{n} &= \frac{c}{8\pi}\sqrt{\frac{\mu}{\varepsilon}}\,\mathrm{Re}\,\boldsymbol{H}_\tau^*(z_0)\Omega^+N_0\Omega\boldsymbol{H}_\tau(z_0)\\
&= \frac{c}{8\pi}\sqrt{\frac{\mu}{\varepsilon}}\,\mathrm{Re}[\boldsymbol{H}_\tau^*(z_0)N_0\boldsymbol{H}_\tau(z_0) + \mathrm{i}R(z)]\\
&= \frac{c}{8\pi}\sqrt{\frac{\mu}{\varepsilon}}\,\mathrm{Re}\,\boldsymbol{H}_\tau^*(z_0)N_0\boldsymbol{H}_\tau(z_0)
\end{aligned}$$

where $R(z)$ contains pure real terms depending on z. It is obvious that the value $\langle\boldsymbol{P}\rangle\cdot\boldsymbol{n}$ is conserved in the $\boldsymbol{n}$-direction (even for the case when N_0 is non-Hermitian, $N_0^+ \neq N_0$).

We shall show later that either $N_0 = \pm I$ ($\mathrm{tr}\,N_0 = \pm 2$, these are discrete roots of the projector I) or $N_0 = \pm(I - 2\boldsymbol{S}\otimes\boldsymbol{C})$ ($\mathrm{tr}\,N_0 = 0$, these are continuous sets of roots of the projector I), where $\boldsymbol{S}$ and $\boldsymbol{C}$ are, in general, complex vectors satisfying the conditions

$$\boldsymbol{S}\cdot\boldsymbol{C} = 1 \qquad \boldsymbol{S}\cdot\boldsymbol{n} = \boldsymbol{C}\cdot\boldsymbol{n} = 0.$$

Indeed, introducing the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ which form the orthonormal basis together with $\boldsymbol{n}$ ($\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{n}\boldsymbol{e}_1 = \boldsymbol{n}\boldsymbol{e}_2 = 0$, $[\boldsymbol{e}_1\boldsymbol{e}_2] = \boldsymbol{n}$) we represent N_0 in the form

$$N_0 = a\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + b\boldsymbol{e}_1\otimes\boldsymbol{e}_2 + c\boldsymbol{e}_2\otimes\boldsymbol{e}_1 + d\boldsymbol{e}_2\otimes\boldsymbol{e}_2$$

where a, b, c, d are unknown coefficients. Then, from the involution condition $N_0^2 = I = \boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \boldsymbol{e}_2\otimes\boldsymbol{e}_2$, one can obtain the system of algebraic equations for a, b, c, d:

$$\begin{aligned}
&a^2 + bc = 1 \qquad bc + d^2 = 1\\
&b(a + d) = 0 \qquad c(a + d) = 0.
\end{aligned}$$

Two different cases are possible, $a \neq -d$ and $a = -d$. For the first, $b = c = 0$ and $a = d = \pm 1$. Then $N_0 = \pm \boldsymbol{I}$. For the second, the equation $a = -d = \sqrt{1-bc}$ is valid, b and c being arbitrary complex numbers. Then

$$N_0 = \sqrt{1-bc}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + b\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + c\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 \qquad \operatorname{tr} N_0 = 0. \tag{10}$$

Let us introduce a tensor $N_1 = \boldsymbol{I} - N_0$, $\operatorname{tr} N_1 = 2$. Considering the tensor $\overline{N}_1$ adjoined to the tensor N_1 [4, 5] and taking account of (10) we conclude that $\overline{N}_1 = 0$. This means that the tensor N_1 is a diad which can be written as $N_1 = 2\boldsymbol{S} \otimes \boldsymbol{C}$. The condition $\operatorname{tr} N_1 = 2$ requires $\boldsymbol{S} \cdot \boldsymbol{C} = 1$. Thus for the second case ($a = -d$)

$$N_0 = \boldsymbol{I} - 2\boldsymbol{S} \otimes \boldsymbol{C} \qquad \boldsymbol{S} \cdot \boldsymbol{C} = 1 \qquad \boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0. \tag{11}$$

Let us discuss the results obtained above. Substituting $N_0 = \pm \boldsymbol{I}$ in (9) it is not difficult to understand that discrete solutions of equation (7) describe usual waves which run in directions $\pm \boldsymbol{n}$, respectively, their polarization being wholly determined by an initial vector $\boldsymbol{H}_\tau(z_0)$. The solutions (11) are more interesting. Here we dwell on a particular case

$$\boldsymbol{S} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \mathrm{i}\alpha \boldsymbol{e}_2) \qquad \boldsymbol{C} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_1 + \frac{\mathrm{i}}{\alpha}\boldsymbol{e}_2\right) \tag{12}$$

where α is a real parameter, not considering some other particular cases (e.g. $N_0 = \boldsymbol{I} - 2\boldsymbol{S} \otimes \boldsymbol{S}$, $\boldsymbol{S} = \boldsymbol{S}^*$, $\boldsymbol{S}^2 = 1$, when N_0 is a reflection operator, or $N_0 = \boldsymbol{I} - 2\boldsymbol{S} \otimes \boldsymbol{S}^*$, $|\boldsymbol{S}|^2 = 1$ and so on). Then

$$N_0 = -\mathrm{i}\left(\frac{1}{\alpha}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \alpha \boldsymbol{e}_2 \otimes \boldsymbol{e}_1\right)$$

and

$$\Omega(z - z_0) = \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{I}\cos[k\sqrt{\varepsilon\mu}(z - z_0)] + \left(\frac{1}{\alpha}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \alpha \boldsymbol{e}_2 \otimes \boldsymbol{e}_1\right)\sin[k\sqrt{\varepsilon\mu}(z - z_0)]. \tag{13}$$

To understand how, in the case under consideration, the evolution of $\boldsymbol{H}_\tau(z)$ depending on z takes place we act on various initial vectors $\boldsymbol{H}_\tau(z_0)$ by the operator $\Omega(z - z_0)$ (13). In particular, if $\boldsymbol{H}_\tau(z_0) = h\boldsymbol{e}_1$ where h is some coefficient of proportionality then

$$\boldsymbol{H}_\tau(z) = h\{\boldsymbol{e}_1 \cos[k\sqrt{\varepsilon\mu}(z - z_0)] - \alpha \boldsymbol{e}_2 \sin[k\sqrt{\varepsilon\mu}(z - z_0)]\} \tag{14}$$

and if $\boldsymbol{H}_\tau(z_0) = -h\alpha\boldsymbol{e}_2$ then

$$\boldsymbol{H}_\tau(z) = h\{-\alpha \boldsymbol{e}_2 \cos[k\sqrt{\varepsilon\mu}(z - z_0)] - \boldsymbol{e}_1 \sin[k\sqrt{\varepsilon\mu}(z - z_0)]\}. \tag{15}$$

It is obvious from (14) and (15) that the extreme point of the vector $\boldsymbol{H}_\tau(z)$ in a projection on a plane perpendicular to $\boldsymbol{n}$ draws an ellipse as the coordinate z rises, the direction of traversal being clockwise. The semi-axes of this ellipse are directed along $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, and its eccentricity is $\sqrt{1-(1/\alpha)^2}$ for $\alpha > 1$ and $\sqrt{1-\alpha^2}$ for $\alpha < 1$. However, all the vectors $\boldsymbol{H}_\tau(z, t)$ depending on time t vary synchronously. Therefore, the wave under consideration is in essence standing.

Now we consider the set of vectors $\boldsymbol{H}_\tau(z, t_0)$ (14) at some fixed time $t = t_0$, the initial points of these vectors being on the coordinate axis and coordinate z characterizing their positions. It is obvious that the vectors $\boldsymbol{H}_\tau(z, t_0)$ form the surface of an elliptical helicoid, and their extreme points lay on some twisted helix. It is important to study the differential characteristics of this helix: curvature, torsion, Darboux vector and so on. Such an analysis is useful in understanding the behaviour of wave surfaces in the three-dimensional space and can be applied for consideration of the operator synthesis of non-harmonic fields [33]. The curve under consideration can be described by the vector function (see also (14))

$$\boldsymbol{r}(z) = h(\boldsymbol{e}_1 \cos kz - \boldsymbol{e}_2\alpha \sin kz) + \boldsymbol{n}z. \tag{16}$$

To simplify calculations we assume $z_0 = 0$ and $\varepsilon = \mu = 1$. Then the derivatives of this function and products of the derivatives are

$$\begin{aligned}
&\boldsymbol{r}' = kh(-\boldsymbol{e}_1 \sin kz - \boldsymbol{e}_2\alpha \cos kz) + \boldsymbol{n}\\
&\boldsymbol{r}'' = k^2h(-\boldsymbol{e}_1 \cos kz + \boldsymbol{e}_2\alpha \sin kz)\\
&\boldsymbol{r}''' = k^3h(\boldsymbol{e}_1 \sin kz + \boldsymbol{e}_2\alpha \cos kz)\\
&|\boldsymbol{r}'| = \sqrt{k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1}\\
&[\boldsymbol{r}' \cdot \boldsymbol{r}''] = k^2h(-\boldsymbol{e}_1\alpha \sin kz - \boldsymbol{e}_2 \cos kz - \boldsymbol{n}kh\alpha)\\
&|[\boldsymbol{r}' \cdot \boldsymbol{r}'']| = k^2h\sqrt{\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2}\\
&[\boldsymbol{r}' \cdot \boldsymbol{r}'']\boldsymbol{r}''' = -k^5h^2\alpha.
\end{aligned} \tag{17}$$

Using the basic definitions of differential geometry from [36] and taking into account equations (17), we find the unit tangent vector

$$\boldsymbol{T} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{-kh(\boldsymbol{e}_1 \sin kz + \boldsymbol{e}_2\alpha \cos kz) + \boldsymbol{n}}{\sqrt{k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1}} \tag{18}$$

the binormal

$$\boldsymbol{B} = \frac{[\boldsymbol{r}' \cdot \boldsymbol{r}'']}{|[\boldsymbol{r}' \cdot \boldsymbol{r}'']|} = -\frac{\boldsymbol{e}_1\alpha \sin kz + \boldsymbol{e}_2 \cos kz + \boldsymbol{n}kh\alpha}{\sqrt{\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2}} \tag{19}$$

the curvature

$$\sigma = \frac{|[\boldsymbol{r}' \cdot \boldsymbol{r}'']|}{|\boldsymbol{r}'|^3} = \frac{k^2h\sqrt{\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2}}{(\sqrt{k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1})^3} \tag{20}$$

the torsion

$$\tau = \frac{[\boldsymbol{r}' \cdot \boldsymbol{r}'']\boldsymbol{r}'''}{|[\boldsymbol{r}' \cdot \boldsymbol{r}'']|^2} = -\frac{k\alpha}{\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2} \tag{21}$$

and, finally, the Darboux vector

$$\begin{aligned}
\boldsymbol{\delta} = \tau\boldsymbol{T} + \sigma\boldsymbol{B} = {}& \frac{k}{\sqrt{k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1}}\left[\frac{k\alpha h(\boldsymbol{e}_1 \sin kz + \boldsymbol{e}_2\alpha \cos kz) - \alpha\boldsymbol{n}}{\sqrt{\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2}}\right.\\
&\left. - \frac{kh(\boldsymbol{e}_1\alpha \sin kz + \boldsymbol{e}_2 \cos kz + \boldsymbol{n}kh\alpha)}{k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1}\right].
\end{aligned} \tag{22}$$

It is known that the Darboux vector of a spatial curve is nothing less than the angular-velocity vector of the moving trihedral $\boldsymbol{TNB}$ ($\boldsymbol{N}$ is the principal normal, $\boldsymbol{N} = [\boldsymbol{B} \cdot \boldsymbol{T}]$) [36]. In the case $\alpha \neq 1$, $\boldsymbol{\delta}$ is not parallel to $\boldsymbol{n}$ and this vector varies in a fairly quaint way, when the coordinate z increases. Its magnitude is

$$\begin{aligned}
|\boldsymbol{\delta}| = k[&k^2h^2(\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2)(k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1)^{-3}\\
&+\alpha^2(\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2)^{-2}]^{1/2}
\end{aligned}$$

and its projection on $\boldsymbol{n}$ is

$$\begin{aligned}
\boldsymbol{\delta n} = {}& -k\alpha[k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1]^{-1/2}\\
&\times\left[\frac{1}{\alpha^2 \sin^2 kz + \cos^2 kz + k^2h^2\alpha^2} + \frac{k^2h^2}{k^2h^2(\sin^2 kz + \alpha^2 \cos^2 kz) + 1}\right].
\end{aligned} \tag{23}$$

If we set the start point of all the vectors $\boldsymbol{\delta}$, corresponding to the different values of z, in an origin point O, then these vectors will lay on the surface of some cone, but this cone is not circular in the general case.

Relations (18)–(23) are simplified to a great extent when $\alpha = 1$ (i.e. when the vectors $\boldsymbol{H}_\tau(z, t_0)$ form a circular helicoid). In particular, the curvature and the torsion in this case are

$$\sigma = \frac{k^2 h}{k^2 h^2 + 1} \qquad \tau = -\frac{k}{k^2 h^2 + 1} \tag{24}$$

and do not depend on z, and the Darboux vector is

$$\boldsymbol{\delta} = -\frac{k}{\sqrt{k^2 h^2 + 1}} \boldsymbol{n}. \tag{25}$$

At $\alpha = 1$, the vector $\boldsymbol{\delta}$ is a constant vector and parallel to $\boldsymbol{n}$ for all z.

Thus at $\alpha = 1$ the direction of the Darboux vector coincides with the direction of the wave normal $\boldsymbol{n}$. This fact will be easily understood if one takes into account the fact that both $\boldsymbol{H}_\tau(z, t_0)$ and the moving trihedral $\boldsymbol{TNB}$ rotates around the coordinate axis as one rigid construction in this case. However, the vector $\boldsymbol{H}_\tau(z, t_0)$ rotates with constant angular velocity $\boldsymbol{\omega}$, and the direction of $\boldsymbol{\omega}$ coincides with the direction of $\boldsymbol{n}$. Therefore, the Darboux vector for the moving trihedral is parallel to $\boldsymbol{n}$. In this case the vectors $\boldsymbol{S}$ and $\boldsymbol{C}$ (12) are circular ($\boldsymbol{S}^2 = \boldsymbol{C}^2 = 0$, $\boldsymbol{S} \cdot \boldsymbol{C} = 1$), the refractive index tensor takes the form of $N = \mathrm{i}\sqrt{\varepsilon\mu}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_1 \otimes \boldsymbol{e}_2) = \mathrm{i}\sqrt{\varepsilon\mu}\boldsymbol{n}^\times$ and the evolutional operator $\Omega(z - z_0)$ is a rotation operator (versor). At $\alpha = 1$ the energy flux $\boldsymbol{P}$ is identically equal to zero for any z and t, and the electric and magnetic fields are parallel. The cases when $\boldsymbol{E} \parallel \boldsymbol{H}$ have been analysed in [38].

The consideration carried out above shows the important role of the concept of the generalized helix and, in particular, the conic helix which appears in absorbing isotropic media described by complex ε and μ.

3. Involutive operators in acoustics

The propagation of elastic waves in the anisotropic medium which is characterized by a tensor of elastic modules c_{iklm} and density ρ is described by Christoffel's equation [5]

$$c_{iklm} \frac{\partial^2 u_m}{\partial x_k \partial x_l} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{26}$$

where $\boldsymbol{u}(\boldsymbol{r}, t) = (u_i(\boldsymbol{r}, t))$ are shift vectors of medium points. We consider the plane wave solutions of this equation

$$\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{u}(z) \exp(-\mathrm{i}\omega t)$$

where ω is wave frequency, $z = \boldsymbol{nr}$, $\boldsymbol{n}$ is the unit vector of the wave normal. Then in (26) the derivative $\partial/\partial t$ is replaced by $-\mathrm{i}\omega$ and the derivative $\partial/\partial x_i$ is replaced by $n_i(\mathrm{d}/\mathrm{d}z)$. We obtain

$$c_{iklm} n_k n_l \frac{\mathrm{d}^2 u_m}{\mathrm{d}z^2} = -\rho\omega^2 u_i$$

or, in index free notation,

$$\Lambda \frac{\mathrm{d}^2 \boldsymbol{u}}{\mathrm{d}z^2} = -\omega^2 \boldsymbol{u}. \tag{27}$$

In equation (27) the reduced Green–Christoffel's tensor $\Lambda = (\Lambda_{im}) = (\lambda_{iklm} n_k n_l)$, $\lambda_{iklm} = c_{iklm}/\rho$ is introduced [5]. This tensor is positive definite for any directions of the normal $\boldsymbol{n}$ and therefore always has the inverse tensor Λ^{-1}. We represent the coordinate dependence $\boldsymbol{u}(z)$ of the vector $\boldsymbol{u}(\boldsymbol{r}, t)$ in the form of

$$\boldsymbol{u}(z) = \exp[\mathrm{i}\omega N(z - z_0)]\boldsymbol{u}(z_0). \tag{28}$$

In (28) the second rank tensor of slowness N is introduced and the value of vector $\boldsymbol{u}$ in the initial point z_0 is assumed to be given. Substituting (28) into (27) and taking into account the arbitrariness of $\boldsymbol{u}(z_0)$ we obtain the following dispersion equation determining the tensor N

$$\Lambda N^2 = 1$$

where 1 is the unit tensor of the three-dimensional space, hence ($|\Lambda| \neq 0$)

$$N = \sqrt{\Lambda^{-1}}. \tag{29}$$

Equations (29) and (28) wholly describe propagation of the plane elastic waves in anisotropic media. We apply the obtained above relations for isotropic materials. For these materials the reduced the Green–Christoffel's tensor has the form

$$\Lambda = (\lambda_{iklm} n_k n_l) = \lambda_{44} 1 + (\lambda_{11} - \lambda_{44})\boldsymbol{n} \otimes \boldsymbol{n} = \lambda_{44} \boldsymbol{I} + \lambda_{11} \boldsymbol{n} \otimes \boldsymbol{n}$$

where λ_{ij} $(i, j = 1, \ldots, 6)$ is the brief notation of elastic modules [5, 37], $\boldsymbol{I} = -\boldsymbol{n}^{\times 2} = 1 - \boldsymbol{n} \otimes \boldsymbol{n}$ is the projector on the plane perpendicular to $\boldsymbol{n}$, and the inequality

$$\lambda_{11} \neq \lambda_{44} \tag{30}$$

always holds. Then

$$N = \sqrt{\Lambda^{-1}} = \lambda_{44}^{-1/2} \sqrt{\boldsymbol{I}} + \lambda_{11}^{-1/2} \boldsymbol{n} \otimes \boldsymbol{n} = \frac{1}{v_{\mathrm{tr}}} \sqrt{\boldsymbol{I}} + \frac{1}{v_{\mathrm{l}}} \boldsymbol{n} \otimes \boldsymbol{n} \tag{31}$$

where $v_{\mathrm{tr}} = \lambda_{44}^{1/2}$ and $v_{\mathrm{l}} = \lambda_{11}^{1/2}$ are the velocities of transverse and longitudinal elastic waves in the isotropic medium, respectively. Note that in (31) the square root just from the projector $\boldsymbol{I}$ on the two-dimensional plane of three-dimensional space is included. This circumstance is explained by the inequality (30) and twofold degeneration of the velocities of transverse elastic waves in the isotropic medium. The branches of square root $\sqrt{\boldsymbol{I}}$ (involutions) was considered in detail in section 2. They can be represented in the general form

$$\sqrt{\boldsymbol{I}} = \pm \boldsymbol{I}$$
$$\sqrt{\boldsymbol{I}} = \pm(\boldsymbol{I} - 2\boldsymbol{S} \otimes \boldsymbol{C}) \tag{32}$$

where $\boldsymbol{S}$ and $\boldsymbol{C}$ are complex vectors yielding to the conditions

$$\boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0 \tag{33}$$

$$\boldsymbol{S} \cdot \boldsymbol{C} = 1. \tag{34}$$

Therefore, the evolution of the transverse components of the shift vector $\boldsymbol{u}$ of the plane wave in acoustics is completely analogous with the evolution of the transverse electric and magnetic field vectors in optics.

Now we consider the case when three phase velocities of the isonormal elastic waves coincide. In particular, the case $v_{\mathrm{tr}} = v_{\mathrm{l}}$ is realized for some classes of transversely isotropic media in some directions of the normal $\boldsymbol{n}$ [39]. It does not contradict the condition of positive definiteness of the potential energy of elastic deformations. Then the equation

$$N = \frac{1}{v} \sqrt{1}$$

holds and in the expression for N the square root from the unit tensor of the three-dimensional space (three-dimensional involutive operators) is included. It can be shown that all the branches of $\sqrt{1}$ are determined by equalities

$$\sqrt{1} = \pm 1$$
$$\sqrt{1} = \pm(1 - 2\boldsymbol{S} \otimes \boldsymbol{C})$$

where $\boldsymbol{S}$ and $\boldsymbol{C}$ are, in general, the complex vectors of the three-dimensional space yielding to condition (34), $\boldsymbol{S} \cdot \boldsymbol{C} = 1$ (but not necessarily (33)). In this case the behaviour of the vector $\boldsymbol{u}(z)$ when z arises is more complex than in case (32) and we intend to investigate this in the future.

4. Conclusion

From the evolutional solutions of Maxwell's equations for the plane wave fields in the isotropic medium it follows that the refractive index N and impedance γ of such fields are operators of the three-dimensional spaces consisting of pairs of the scalar and tensor cofactors, $N = \sqrt{\varepsilon\mu}\sqrt{1 - \boldsymbol{n} \otimes \boldsymbol{n}}$ and $\gamma = \sqrt{\mu/\varepsilon}\sqrt{1 - \boldsymbol{n} \otimes \boldsymbol{n}}$. Cofactors $\sqrt{\varepsilon\mu}$ and $\sqrt{\mu/\varepsilon}$ characterize the refraction and wave impedance of the substance and branches of the tensor $\sqrt{1 - \boldsymbol{n} \otimes \boldsymbol{n}}$ being equal to $\pm(1 - \boldsymbol{n} \otimes \boldsymbol{n})$ and $\pm(1 - \boldsymbol{n} \otimes \boldsymbol{n} - 2\boldsymbol{S} \otimes \boldsymbol{C})$, where $\boldsymbol{S}$ and $\boldsymbol{C}$ are complex vectors ($\boldsymbol{S} \cdot \boldsymbol{C} = 1$, $\boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0$), describe the wave polarization properties of the medium including the vacuum too. This root from the projective operator of the wavefront subspace orthogonal to the wave normal $\boldsymbol{n}$ has an infinite set of branches which are reflectional and rotational isometries of the field. These isometries are involutive operators which locally characterize the symmetry of the electromagnetic field. This set of operators form continuous groups of transformations known in mathematical literature as the kaleidoscopic Coxeter's groups. These groups generate the involutive global Lie groups of the wave equation solutions and simultaneously characterize the symmetry of equations and solutions. The tensor $\sqrt{1 - \boldsymbol{n} \otimes \boldsymbol{n}}$ is in essence the root from the metric tensor of the two-dimensional (adjoined) subspace of the wavefront immersed in the three-dimensional space, the third dimension of which is counted out along the normal $\boldsymbol{n}$ (direction of the photon momentum). Our exponential solutions are the mathematical expressions of Huygens principle for polarized light. We saw that for the continuous Lie group of solutions of the one-dimensional Helmholtz equation there corresponds a manifold of elliptical helices in three-dimensional space. The theory of the mobile Frenet–Serret trihedral has been applied for finding curvatures and torsions of these lines and Darboux vectors. It has been shown that in the case of circular lines the Darboux vectors are directed along the light beam.

In the acoustics of isotropic media the tensor of slowness is represented as a linear combination of the root $\sqrt{1 - \boldsymbol{n} \otimes \boldsymbol{n}}$ and the diad $\boldsymbol{n} \otimes \boldsymbol{n}$. Such a structure of this tensor is the result of non-coinciding velocities of the longitudinal and transverse isonormal waves and twofold degeneracy of the transverse wave velocities. The evolutions of the transverse components of fields in the optics and acoustics of isotropic media are described, therefore, by the same generator $\sqrt{1 - \boldsymbol{n} \otimes \boldsymbol{n}}$ and are wholly analogous.

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